Mathematical Problems in Industry 2022
Problem Statements

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1 Commitment Planning for Drawdown Vehicles in a Stochastic Setting

Over the past several decades investments in illiquid asset classes such as venture capital, private equity and private credit (hereafer we'll refer to these asset classes collectively as "alternatives") have played an increasingly large role in the portfolios of many institutional investors. For example based on S&P Global SNL Financial data, GSAM estimates that private equity accounted for $122bn of the assets on US Life Insurers' balance sheets as of year end 2021 compared to $8.6bn as of year end 2006. Unlike investing in liquid asset classes such as public equity, investing in alternatives is typically implemented through commingled limited partnership (LP) funds. These structures introduce implementation challenges not present in the context of liquid public investments. Per [Takahashi and Alexander(2002)]:

> "The funds are raised every few years on a blind pool basis by general partners who actively invest, manage and harvest portfolio investments. At the onset of the partnership, investors commit capital that gets drawn down over several years by the general partner. The uncertain schedule of drawdowns, unknowable changes in the valuation of the partnership's investments, and unpredictable distributions of cash or securities to the limited partners combine to make it difficult to predict accurately the future value of partnership interests."

Therefore in addition to the standard problem of deciding their strategic asset allocation (SAA), when investing in alternatives the investor is also faced with the non-trivial problem of designing a multi-year commitment plan (CP) that will achieve a desired alternatives allocation. The LPs (investor clients whose financial interests are to be served) make capital commitments to the general partners (GPs) of the fund, the GPs then call this capital over a period of years, invest the capital and eventually unwind the investments and distribute cash flows back to the LPs. [Takahashi and Alexander(2002)] introduced a commitment planning model (often referred to as the "Yale model") which will be described in the subsequent section. While the Yale model offers a very useful and intuitive framework for projecting cash flows of alternative assets, one drawback it suffers from is the assumption of a constant deterministic rate of return for each alternative asset class. Under this assumption a CP can be easily designed that reaches a given target allocation for each alternatives asset class (in steady state). However in the stochastic context, where returns for each asset class are stochastic the CP may need to be dynamically adapted over time. A balance must also be sought between the complexity of the commitment plan (which will generally have to be something simple enough to describe and put into writing within an investment policy statement) and the expected ability of the CP to quickly ramp up to and track the desired asset allocation.

The **Yale model** [Takahashi and Alexander(2002)] keeps track of the following variables for a given fund as a function of discrete time period t, which we take as an integer measuring year since the entrance of the investor:

Variable	Name	Description
C	Capital contributions	Capital called by the GP (\$)
D	Capital distributions	Capital distributed by the GP (\$)
NAV	Net Asset Value	Estimated fair market value of the fund assets (\$)

The model is calibrated using the following parameters

Variable	Name	Description
$\mathrm{RC}_{(t)}$	Rate of contribution	Rate of capital calls by the GP in year t (%)
CC	Capital commitments (\$)	LP's total commitment to GP (\$)
L	Life of fund	Total number of years fund is active (years)
B	Bow	Factor describing changes in the rate of distribution over time
G	Annual Growth Rate (%)	Assumed rate of return
Y	Yield (%)	Only applicable for yield focused asset classes

It is also convenient to define variables for the following quantities which are derived from above

Variable	Name	Description
PIC	Paid in capital	Sum of capital contributions to date (\$)
$\mathrm{RD}_{(t)}$	Rate of distribution	Rate of capital distributions in year t (%)

The dynamics of the model over discrete time periods of years are governed by the following relationships

$$\mathrm{PIC}_{(t)} = \sum_{i=0}^{t-1} C_{(i)} \tag{1a}$$

$$C_{(t)} = \mathrm{RC}_{(t)} \left(\mathrm{CC} - \mathrm{PIC}_{(t)}\right) \tag{1b}$$

$$D_{(t)} = \mathrm{RD}_{(t)} \left(\mathrm{NAV}_{(t-1)}(1+G)\right) \tag{1c}$$

$$\mathrm{RD}_{(t)} = \max\left(Y, (t/L)^B\right) \tag{1d}$$

$$\mathrm{NAV}_{(t)} = \left(\mathrm{NAV}_{(t-1)}(1+G)\right) + C_{(t)} - D_{(t)} \tag{1e}$$

The first equation just expresses the total capital already paid in by the LP by year t as a simple sum of the capital contributions made in each previous year. The second equation expresses the amount of capital to be contributed in year t as the remaining committed capital of the LP multiplied by the contribution rate set by the GP. For strategic planning purposes, this contribution rate $\mathrm{RC}_{(t)}$ will be specified empirically from historical data. The third equation expresses the distribution received in year t as the net asset value from the previous year augmented by its growth over the subsequent year, multiplied by the rate at which the fund sheds distributions. The fourth equation specifies the distribution rate in terms of a power law growth from the inception of the fund to its closing, with a floor set by the underlying yield when relevant. Note the fund will be entirely liquidated and returned to the investors at the end of the fund's life ($\mathrm{RD}_{(L)} = 1$). The last equation expresses the updated net asset value of the fund by combining the growth from the previous year with the net inflow of capital into the fund from the LP.

1.1 Commitment Planning Using the Yale Model

Consider the problem faced by an investor who has, via some process external to the considerations here, arrived at a desired target allocation level T (\$) for some asset class (e.g. Buyout). In other words, T is the level of investment (exposure) $\mathrm{NAV}_{(t)}$ desired in the asset class. Further let's assume that investments in the asset class in question are typically implemented using the limited partnership structure described above.

The investor must figure out what level of annual commitments to Buyout funds will lead to the desired target allocation level. The Yale model may be used for this purpose.

In reality, the investor client will want to diversify their alternatives investment into funds raised in various years, rather than a single fund opportunity. Following the industry's language, we refer to the year index in which a fund is raised as a vintage. If we simply assume the funds of each vintage have the same growth rate G, then the Yale model is simply extended by indexing all state variables by the vintage v of the fund they reference; this is done in the superscript. Thus, Equations (1) are now extended by consideration of various fund vintages to read:

$$\mathrm{PIC}_{(t)}^{(v)} = \sum_{i=0}^{t-1} C_{(i)}^{(v)} \tag{2a}$$

$$C_{(t)}^{(v)} = \mathrm{RC}_{(t-v)} \left(\mathrm{CC}^{(v)} - \mathrm{PIC}_{(t)}^{(v)} \right) \tag{2b}$$

$$D_{(t)}^{(v)} = \mathrm{RD}_{(t-v)} \left(\mathrm{NAV}_{(t-1)}^{(v)} (1+G) \right) \tag{2c}$$

$$\mathrm{RD}_{(t)} = \max \left(Y, (t/L)^B \right) \tag{2d}$$

$$\mathrm{NAV}_{(t)}^{(v)} = \left(\mathrm{NAV}_{(t-1)}^{(v)} (1+G) \right) + C_{(t)}^{(v)} - D_{(t)}^{(v)} \tag{2e}$$

For this idealized setting, one can solve in closed form for a commitment plan that exactly maintains the desired target allocation $\mathrm{NAV}_{(t)} = T$ for all $t \geq 0$, where

$$\mathrm{NAV}_{(t)} = \sum_{v=0}^{t-1} \mathrm{NAV}_{t}^{(v)} \tag{3}$$

This solution will be developed as a warmup exercise for the main problem, which we specify next.

1.2 Stochastic Extension Single Asset Class

In reality, we do not expect the growth rate of a fund to be a simple known constant. Rather, we can expect at least two types of uncertainty. One is general dynamical stochasticity, so that the growth rate of a given fund of a given vintage has some independent noise in each time period, with some specified probability distribution. The second is variability across vintages. For example, the growth rate of a fund with a vintage of 2021 might perform quite differently from one with a vintage of 2019. To account for both of these uncertainties, we replace the constant deterministic growth rate G in Eq. (2) with a rate $G_{(t)}^{(v)}$ with random variations both across vintages and time. We can no longer expect to be able to design a strategy that meets the ideal target allocation $\mathrm{NAV}_{(t)} = T$ precisely, so the goal now is to design a commitment plan such that the tracking error between the realized allocation $\mathrm{NAV}_{(t)}$ and the desired allocation T is as small as possible in some probabilistic sense. The commitment plan could take the form of a relationship

$$\mathrm{CC}^{(v)} = f \left(\mathrm{CC}^{(0)}, \ldots, \mathrm{CC}^{(v-1)}, \mathrm{NAV}_{(v-1)}^{(0)}, \ldots, \mathrm{NAV}_{(v-1)}^{(v-1)} \right) \tag{4}$$

between the capital committed to the vintage of the funds being currently planted and the capital commitments to previous fund vintages together with their associated valuations. Furthermore there is a qualitative balance to be struck between the complexity of the functional form of f and its ability to achieve the tracking error minimization goal stated above. The commitment plan (4) would naturally also be expected to depend on the parameters describing the randomness of $G_{(t)}^{(v)}$.

1.3 Stochastic Extension Multi-Asset Portfolio

If time permits, the analysis could be extended to multi-asset portfolios where the investor has a target allocation for several illiquid asset classes, as well as several liquid asset classes. The Yale model parametrizations for each illiquid asset class may be distinct. Furthermore the growth rates for each asset class have some non-trivial covariance structure.

References

[Takahashi and Alexander(2002)] D. Takahashi and S. Alexander. Illiquid alternative asset fund modeling. *The Journal of Portfolio Management*, 28(2):90–100, 2002.